

Inter-class orthogonal main effect plans for asymmetrical experiments.

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Abstract : In this paper we construct ‘inter-class orthogonal’ main effect plans (MEP) for asymmetrical experiments. In such a plan, a factor is orthogonal to all others except possibly the ones in its own class. We have also defined the concept of “partial orthogonality” between a pair of factors. In many of our plans, “partial orthogonality” has been achieved when (total) orthogonality is not possible due to divisibility or any other restriction.

We present a method of obtaining ‘inter-class orthogonal’ MEPs. Using this method and also a method of ‘cut and paste’ we have obtained several series of ‘inter-class orthogonal’ MEPs. Interestingly some of these happen to be orthogonal MEP (OMEP); for example we have constructed an OMEP for a 3^{30} experiment on 64 runs. Further, many of the ‘inter-class orthogonal’ MEPs are ‘almost orthogonal’ in the sense that each factor is orthogonal to all others except possibly one. In many of the other MEPs factors are “orthogonal through another factor”, thus leading to simplification in the analysis. Plans of small size (≤ 15 runs) are also constructed by ad-hoc methods.

Finally, we present a user-friendly computational method for analysing data obtained from any general factorial design.

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1 Introduction

In many industrial experiments like screening experiments, often the interest lies only in the main effects of factors. The wide use of orthogonal main effect plans (OMEP) for such experiments is due to their orthogonality property which ensures uncorrelated and hence most precise estimation of every main effect contrast of every factor, apart from providing great simplicity in analysis, as is well-known.

However, orthogonality requires certain divisibility conditions and so an OMEP for an asymmetrical experiment often requires a large number of runs. [See Dey and Mukherjee (1999) and Hedayat, Sloan, and Stufken (1999) for details]. The proportional frequency (PF) plans of Addelman (1962), as we know, are OMEPs with possibly unequal replications for one or more factors, thus requiring weaker conditions for existence. However, very few unequally replicated PF plans are known, apart from Stark’s (1964) plan for a 3^7 experiment on 16 runs. Thus, the problem of availability of an OMEP with not-too-large run size remains. In such situations, therefore, the question arises whether with a smaller run size one can find an alternative plan - something not as good as an OMEP but not too bad either.

Of late, departure from full orthogonality has been investigated in the context of main effect plans (MEPs). In the “nearly orthogonal” plans of Wang and Wu (1992) factors are allowed to be non-orthogonal to a few of the other factors. Subsequently, other nearly orthogonal MEPs having interesting combinatorial properties have been proposed and studied by others like Nguyen (1996), Ma, Fang and Liski (2000), Huang, Wu and Yen (1992) and Xu (2002).

Why do we look for “near orthogonality”? Why can’t we go far away and use a fully non-orthogonal plan ? If we are willing to use non-orthogonal plans, we would have tremendous flexibility. We could, for instance, make a 2^4 experiment on 5 runs (instead of 8) [see plan $A_5(4)$ of example 2.1] or a 3^5 experiment on 12 runs (instead of 16) [see Plan $A_{12}(4)$ in section 5]. One hurdle to the usability of such plans is the complexity in the data analysis. The reduction in the precision is, of course, another problem.

In the present paper our main aim is to provide main effect plans (MEPs) for asymmetrical experiments with small run size, deviating “as little as possible” from the desirable properties like orthogonality and/or equal replications, so that analysis remains relatively simple. Specifically, we construct plans satisfying “**inter-class orthogonality**”, in which each factor is possibly non-orthogonal to the members of its own class, but orthogonal to factors of other classes. In the process we have also obtained a series of **orthogonal MEP for a 3^{30} experiment on 64 runs** (see Theorem 3.5). In many of our plans the class size is at most two, so that a factor is orthogonal to all others except possibly one. Among plans of larger class size, many plans satisfy the property that within class factors are “orthogonal through another factor” (in the same class), thus leading to simplification in the analysis. (See Example 2.1 and Theorem 3.3 (c)).

We have also defined the concept of “**partial orthogonality**” between a pair of factors and derived sufficient condition for it. [See definition 2.2, Lemma 2.1 and the discussion thereafter. In many of our plans, “partial orthogonality” has been achieved between one or more pair(s) of factors when (full) orthogonality is not possible due to divisibility or any other restriction.

The definitions along with examples are presented in section 2. In section 3 we construct a few series of “inter-class orthogonal” MEPs for asymmetrical experiments with factors having at most five levels. Using ad-hoc methods we have also constructed MEPs with factors nonorthogonal to one or two factors on at most 15 runs, which are in Section 5. These plans include saturated plans for the following experiments. $4^2.2$, $3^2.2^3$ and $4.3.2^2$ on 8 runs, $5^2.2$ on 10 runs, $4^2.3^2.2$ and 2.3^5 on 12 runs and $5^2.3^2$ on 15 runs. In section 4 we present an user-friendly method of analysis.

We believe that the information presented in Theorem 4.6 and other results in section 4 will help the experimenter to have a clear idea about the efficiencies of the BLUEs of the main effects as well as the amount of computation involved in the analysis of a non-orthogonal plan. These features may be compared with those of other available plans like “plan orthogonal through one factor” or an “inter-class orthogonal plan”.

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2 Inter-class orthogonal main effect plans

Throughout this paper we shall be concerned with main effect plans, that is plans aiming at gaining information only about the main effects, assuming interactions to be negligible. In all plans presented henceforth in this paper, **rows represent factors, while columns represent runs**.

Let us consider a main effect plan (MEP) for an experiment with m factors A, B, \dots on n runs. Suppose the factor A have a levels, B have b levels and so on. Then, the plan will be referred to as an m -factor MEP and will be represented by an $m \times n$ array $\rho(n, m; a, b, \dots)$.

$r_A(i)$ will denote the number of runs in which factor A is at level i , while the vector $r_A = (r_A(1), \dots, r_A(a))$ will be referred to as the **replication vector** of the factor A . For two factors A, B the incidence matrix N_{AB} is the $a \times b$ matrix with the (i, j) th entry $n_{AB}(i, j) =$ **the number of runs in which A is at level i and B is at level j** . Clearly, N_{AA} is a diagonal matrix whose diagonal entries are those of r_A in the same order and will sometimes be denoted by R_A .

Definition 2.1 *Let us consider an m -factor MEP ρ on n runs. Suppose the set of factors of ρ can be divided into several classes in such a way that **every factor is orthogonal to every other from a different class**. Then ρ is called **inter-class orthogonal**. An inter-class orthogonal MEP with m factors divided into p classes and the factors in the i -th class having levels s_{i1}, s_{i2}, \dots on n runs will be denoted by $\rho(n, m; \{s_{11}.s_{12} \dots\}, \{s_{21}.s_{22} \dots\} \dots)$. A plan with at most m factors in a class may be referred to as an **inter-class(m) orthogonal MEP**.*

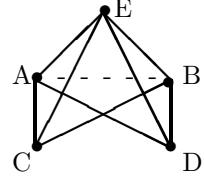
Remark 2.1: Any main-effect plan may be looked upon as an inter-class(m) orthogonal MEP, for some m . For instance, an OMEP may be viewed as an inter-class (1) orthogonal MEP, while an MEP

with p factors of which no one orthogonal to any other is inter-class(p) orthogonal. The plan $L'_{18}(3^4.2^8)$ of Wang and Wu(1992) is an inter-class (8) orthogonal MEP, according to the present terminology, as the 8 two-level factors are mutually non-orthogonal. We see that the term inter-class(m) orthogonal does not always display the exact picture as there may be classes with size much smaller than m , as in the case of $L'_{18}(3^4.2^8)$. This term is informative when the class sizes are close to one another, which is the case for the plans constructed here.

Examples : We now present two inter-class orthogonal plans and along with their graphical representation. Here adjacency represents orthogonality. The interpretation of the dotted line between factors is explained in Remark 2.4.

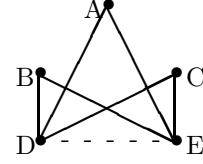
Example 2.1: $A_8(1) = \rho(8, 5; \{3^2\}.\{2^2\}.2)$.

$$A_8(1) = \begin{bmatrix} A & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 2 \\ B & 0 & 0 & 1 & 2 & 2 & 1 & 0 & 0 \\ C & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ D & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ E & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$



Example 2: $A_{12}(1) = \rho(12, 5; \{2.4^2\}.\{3^2\})$.

$$A_{12}(1) = \begin{bmatrix} A & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ B & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ C & 1 & 2 & 3 & 2 & 3 & 0 & 3 & 0 & 1 & 0 & 1 & 2 \\ D & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ E & 0 & 1 & 2 & 2 & 1 & 0 & 0 & 2 & 1 & 1 & 2 & 0 \end{bmatrix}$$



In this equal frequency saturated plan, both the four-level factors B and C form a generalized group divisible design with the levels of the two-level factor A. Between themselves, they form a Balanced incomplete block design (BIBD). The relation between factors D and E is presented in details after Remark 2.4.

The relation between the factors A and B in $A_8(1)$ and factor D and E in $A_{12}(1)$ motivates us to define the concept of partial orthogonality between two factors.

Definition 2.2 We say that the factor A is partially orthogonal (PO) to another factor B if the BLUE of at least one (but not all) main effect contrast of A is orthogonal to the BLUE of every one of B.

Remark 2.2. One can verify that in the MEP presented in (2) of Huang, Wu and Yen (2002), the three-level factors are partially orthogonal to each other. In fact the relation between every pair of three-level factors in that plan just like the relation between A and B of $A_8(1)$. [See Table 2.1 below]. More examples are in Section 3.

Remark 2.3. If A is PO to B, then B is either PO (plan $A_8(1)$) to A or non-orthogonal (plan $A_4(2)$ in section 3) to A. Regarding analysis, however, what matters is whether A and B are mutually orthogonal or not. Thus, partial orthogonality is a feature of estimation and has no role to play in testing of hypothesis.

Remark 2.4. If two factors are partial orthogonal to each other, then in the graphical representation they are joined by dotted lines.

A statement like “A is PO to B” immediately raise the question “which contrast of A is orthogonal to B”? We shall now see how the incidence matrix N_{AB} helps us to find at least partial answer to this question.

How to check orthogonality of a contrast of A to those of B.

We recall the proportional frequency condition of Addelman (1962).

Definition 2.3 [Addelman (1962)] Consider a main effect plan ρ on n runs. Two factors A and B are said to be orthogonal to each other, if the incidence matrix N_{AB} of A and B satisfies the proportional frequency condition (PFC), as stated below.

$$n_{A,B}(i, j) = r_A(i) \cdot r_B(j) / n, \quad i = 1, 2, \dots, a, j = 1, 2, \dots, b. \quad (2.1)$$

We now define PFC between one factor and certain levels of another factor.

Definition 2.4 Consider two factors A and B , with a and b levels respectively, of a main effect plan ρ on n runs.

(a) If a level i of A satisfies

$$n_{A,B}(i, j) = r_A(i) \cdot r_B(j) / n, \quad j = 1, 2, \dots, b, \quad (2.2)$$

then we say that the level i of A satisfies PFC with factor B .

(b) If a pair of levels i and k of A satisfies

$$n_{A,B}(i, j) / r_A(i) = n_{A,B}(k, j) / r_A(k), \quad j = 1, 2, \dots, b, \quad (2.3)$$

then the pair $\{i, k\}$ of levels of A is said to satisfy PFC with factor B .

We use the notation α_i for **the unknown effect of level i of the factor A** , $1 \leq i \leq a$ and similar notation for other factors. Further, $\hat{\alpha}_i - \hat{\alpha}_j$ will denote the BLUE of the contrast $\alpha_i - \alpha_j$. Similar notation for other contrasts.

The proof of the following result is by straightforward verification.

Lemma 2.1 (a) If a level i of A satisfy PFC with B , then the BLUE of the main effect contrast $(a - 1)\alpha_i - \sum_{j \neq i} \alpha_j$ is orthogonal to the BLUEs of all main effect contrasts of B .

(b) If the pair of levels $\{i, k\}$ of A satisfies PFC with B , then the BLUE of the main effect contrast $\alpha_i - \alpha_k$ is orthogonal to the BLUEs of all main effect contrasts of B .

We now illustrate these results with the help of plan $A_8(1)$ and another plan presented later.

Plan $A_8(1)$: We note that factors A and C satisfies PFC (see equation (2.1)) and hence they are mutually orthogonal. Similarly, the pairs $(A, D), (A, E), (B, C), (B, D)$ and (B, E) are also mutually orthogonal. Regarding the pair of factors (A, B) , we see that PFC condition is not satisfied. However, level 0 of A satisfies PFC with factor B , as shown in the table below. Therefore, by (a) of Lemma 2.1 the contrast $2\hat{\alpha}_0 - \hat{\alpha}_1 - \hat{\alpha}_2$ is orthogonal to both the contrasts of B . By the same argument the contrast $2\hat{\beta}_0 - \hat{\beta}_1 - \hat{\beta}_2$ is orthogonal to both the contrasts of A .

Table 2.1

| | | $B \rightarrow$ | | | | | | |
|-------------------|---|-----------------|---|---|------------------|----------------------|-----|-----|
| $A \downarrow$ | | 0 | 1 | 2 | $r_A \downarrow$ | $r_A \cdot r'_B / n$ | | |
| N_{AB} | 0 | 2 | 1 | 1 | 4 | 2 | 1 | 1 |
| | 1 | 1 | 1 | 0 | 2 | 1 | 1/2 | 1/2 |
| | 2 | 1 | 0 | 1 | 2 | 1 | 1/2 | 1/2 |
| $r_B \rightarrow$ | | 4 | 2 | 2 | $n = 8$ | | | |

Plan $A_{12}(1)$: We note that the pair of three-level factors D and E do not satisfy PFC condition. However, levels 0 and 2 of D satisfy PFC with E and so by (b) of Lemma 2.1 the contrast $\hat{\delta}_0 - \hat{\delta}_2$ is orthogonal to both the contrasts of E . By the same argument the contrast $\hat{\epsilon}_1 - \hat{\epsilon}_2$ is orthogonal to both the contrasts of D . In the following table r denotes the constant replication number of D .

Table 2.2

| | | $E \rightarrow$ | | | | | | |
|-------------------|---|-----------------|---|---|------------------|------------|-----|-----|
| $D \downarrow$ | | 0 | 1 | 2 | $r_D \downarrow$ | N_{DE}/r | | |
| N_{DE} | 0 | 2 | 1 | 1 | 4 | 1/2 | 1/4 | 1/4 |
| | 1 | 0 | 2 | 2 | 4 | 0 | 1/2 | 1/2 |
| | 2 | 2 | 1 | 1 | 4 | 1/2 | 1/4 | 1/4 |
| $r_E \rightarrow$ | | 4 | 4 | 4 | $n = 12$ | | | |

Discussion: What is the use of partial orthogonality ? This may be viewed as a “something is better than nothing” approach. If it is not possible to make A and B mutually orthogonal, we may at least make them partially orthogonal, if possible. However, the issue is more complicated, since, to achieve one condition, we may have to sacrifice another. Let us look at the following situations. Consider two factors A and B with a and b levels respectively.

Case 1. ab does not divide n , so that there does not exist any plan in which A and B are mutually orthogonal, each with equal frequency. In case a proportional frequency plan exists, then of course, that is the best option. Suppose such a plan is not known. If we know a plan, say ρ_1 , in which A is partially orthogonal to B , then the experimenter would be happy to be able to estimate at least a few among the main effect contrasts of A with maximum precision. However, this may lead to “too small” a precision for the other contrasts of A . Suppose a plan ρ_2 is also available in which A and B are not partially orthogonal, but all the contrasts of A and B are estimated with “reasonably high” precision. Whether the experimenter would prefer ρ_1 or ρ_2 depends on the importance she attaches to each contrast. [See Remark 3.3].

Case 2. ab divides n , so that orthogonality between A and B is possible. However, in the only available plan (say ρ_1) in which A and B are mutually orthogonal, various other pairs of factors are mutually non-orthogonal. Suppose another plan ρ_2 is also available in which A is only partially orthogonal to B , but several pairs of factors which are mutually non-orthogonal in ρ_1 are orthogonal in ρ_2 . Which plan should the experimenter choose ? Again, The choice depends on the importance attached to different contrasts of different factors. [See Remark 5.1].

We hope that in future more and more nearly orthogonal, inter-class orthogonal and other similar plans will be available and the experimenters will have a wider range of options.

Orthogonality through another factor:

The concept of “orthogonality (between two treatment factors) through a nuisance factor” has been introduced in Morgan and Uddin (1996) in the context of nested row-column designs. In Bagchi (2010) “orthogonality through the block factor (OTB)” is studied in details. This concept can easily be extended to the case when the third factor is also a treatment factor.

Definition 2.5 Consider three factors A , B and C of an MEP. We say that A is **orthogonal to B “through” C** if the incidence matrices N_{AB} , N_{BC} and N_{AC} satisfy the following condition.

$$N_{AC}(R_C)^{-1}N_{CB} = N_{AB}. \quad (2.4)$$

Example 2.1: Consider two MEPs with two- and three-level factors, on 5 runs : $A_5(1) = \rho(5; \{3 \times 2^2\})$ and $A_5(2) = \rho(5, 3; \{2^4\})$.

$$A_5(1) = \begin{bmatrix} A & 0 & 1 & 0 & 1 & 0 \\ B & 0 & 1 & 1 & 0 & 0 \\ C & 0 & 0 & 1 & 1 & 2 \end{bmatrix} \quad A_5(2) = \begin{bmatrix} A & 0 & 0 & 1 & 1 & 0 \\ B & 0 & 1 & 0 & 1 & 0 \\ C & 0 & 1 & 1 & 0 & 0 \\ D & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (2.5)$$

In the plan $A_5(1)$ A is orthogonal to B through C , while in $A_5(2)$ every factor in $\{A, B, C\}$ is orthogonal to every other through D .

For examples of more such plans, see the equations next to (3.14). For analysis see Theorem 4.5. See also Remark 4.5.

3 Construction of inter-class orthogonal plans

Definition 3.1 Consider an MEP $\rho(n, m; a, b, \dots)$. Suppose there exists another MEP $\rho_1(a, l; t_1, t_2, \dots, t_l)$, $(l \geq 2)$, such that

$$\sum_{i=1}^l (t_i - 1) \leq a - 1. \quad (3.6)$$

Then, we construct a new MEP $\tilde{\rho}$ with n runs by replacing the level u of factor A by the u -th column (run) of ρ_1 , for each u , $0 \leq u \leq a - 1$. We say that **the factor A is replaced by a class G_A of l factors related through ρ_1 and ρ_1 will be said to the replacing array for A** . In the same way we can replace two or more factors of a given MEP, through two or more suitable replacing arrays.

We now try to find conditions on the replacing array so that the resultant plan satisfies certain desirable properties.

Lemma 3.1 Consider a set of factors $\mathcal{R} = \{A, B, \dots\}$ of an MEP ρ . Suppose $\tilde{\rho}$ is an MEP obtained from ρ by replacing each factor of \mathcal{R} by a group of factors. More precisely, the factor A (respectively B) is replaced by the class of factors G_A (respectively G_B) related through ρ_A (respectively ρ_B). Then, the following hold.

(a) If A and B are mutually orthogonal (with equal or unequal frequency) in the original plan ρ then every factor in the class G_A is orthogonal to every factor in the class G_B in the derived plan $\tilde{\rho}$, generally with unequal frequency.

(b) In $\tilde{\rho}$ two factors of G_A will be partially (respectively totally) orthogonal if and only if the corresponding rows of ρ_A are partially (respectively totally) orthogonal.

(c) If ρ and each of the replacing arrays ρ_A etc. are saturated, then so is $\tilde{\rho}$.

Proof : We shall prove (a). (b) will follow by similar argument and (c) by straightforward counting.

Proof of (a): Fix a factor, say K of G_A and a factor L of G_B . Let β_s (respectively γ_t) denote the set of runs of ρ_A (respectively ρ_B) in which the level s of K (respectively t of L) appear.

Let $\tilde{N}_{K,L}$, \tilde{r}_K , \tilde{r}_L denote the incidence matrix of K, L and the replication vectors of K and L respectively in $\tilde{\rho}$. Then, the (s, t) th entry of $\tilde{N}_{K,L}$ is given by

$$\tilde{n}_{K,L}(s, t) = \sum_{i \in \beta_s} \sum_{j \in \gamma_t} n_{A,B}(i, j). \quad (3.7)$$

From this, we obtain that for a level s of K ,

$$\tilde{r}_K(s) = \sum_{i \in \beta_s} \sum_t \sum_{j \in \gamma_t} n_{A,B}(i, j) = \sum_{i \in \beta_s} r_A(i). \quad (3.8)$$

Similarly, $\tilde{r}_L(t) = \sum_{j \in \gamma_t} r_B(j)$. But $N_{A,B}, r_A, r_B$ satisfy (2.1) by hypothesis. Combining that with the relations above we see that $\tilde{M}_{K,L}, \tilde{r}_K, \tilde{r}_L$ also satisfy (2.1). \square

We present the well-known definition of an orthogonal array.

Definition 3.2 Let $m, n, t \geq 2$ be integers and $s = (s_1, \dots, s_m)$ be a vector of integers ≥ 2 . Then an orthogonal array of strength t is an $m \times n$ array, with the entries of the i th row coming from a set of s_i symbols satisfying the following. All t -tuples of symbols appear equally often as rows in every $n \times t$ subarray. Such an array is denoted by $OA(m, n, s_1 \times \dots \times s_m, t)$. When $s_1 = s_2 = \dots = s_m = s$, say, this array is represented by $OA(n, m, s, t)$.

Corollary 3.1 Suppose there exists an orthogonal array $OA(n, m, s, 2)$. Suppose further for an integer $k(< m)$, there exist arrays $\rho_i = \rho(s, l_i; t_{i,1}, \dots, t_{i,l_i})$, satisfying $s - 1 \geq \sum_{j=1}^{l_i} (t_{i,j} - 1)$, $i = 1, 2, \dots, k$.

Then, an inter-class orthogonal array $\rho(n, l; \prod_{i=1}^k \{t_{i,1} \times \dots \times t_{i,l_i}\}, s^{m-k})$ exists. Here $l = \sum_{i=1}^k l_i$.

Proof : Let ρ_0 be the orthogonal MEP represented by the given orthogonal array. We replace the i th factor by a group of factors related through ρ_i , $i = 1, \dots, k$ to form a new MRP ρ . Clearly ρ has $l = \sum_{i=1}^k l_i + m - k$ factors. That ρ is interclass orthogonal with the given parameters follows from Lemma 3.1. \square

Examples of replacing arrays with desirable properties:

We have seen that to obtain an useful inter-class orthogonal MEP, one needs replacing arrays with desirable properties. We now present a few such arrays. In each plan, the factors are named as A, B, \dots , in that order. The set of s levels of a factor will be denoted by the set of integers modulo s .

Plan with s runs, two factors with p and q levels, $p + q = s$:

$$A_s(1) = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 & 2 & \dots & p-1 \\ 0 & 1 & \dots & q-1 & 0 & 0 & \dots & 0 \end{bmatrix}. \quad (3.9)$$

Remark 3.1: The BLUE of the contrast $\alpha_i - \alpha_j$ of factor A is orthogonal to the BLUEs of the contrasts of B , for $i \neq j$, $i, j \geq 1$. Similarly, the BLUE of the contrast $\beta_i - \beta_j$ of B is orthogonal to the BLUEs of the contrasts of A , for $i \neq j$, $i, j \geq 1$.

Plans with 4 runs : A plan, say $A_4(1)$ may be obtained by putting $s = 4, p = 2, q = 3$ in (3.9). We now present another plan.

$$A_4(2) = \rho(4; \{3 \times 2\}) = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Remark 3.2 : (a) In $A_4(2)$, (2.2) is satisfied by level 0 of factor A , so that $2\hat{\alpha} - \hat{\alpha}_1 - \hat{\alpha}_2$ is orthogonal to the BLUEs of the main effect contrast of B .

Plans with 5 runs : Two plans, namely $A_5(1) = \rho(5; \{2^2 \times 3\})$ and $A_5(2) = \rho(5; \{2^4\})$ are presented in Example 2.1. Two other plans $A_5(3)$ and $A_5(4)$ are obtained from (3.9) by putting $s = 5, p = 4, q = 2$ and $s = 5, p = q = 3$ respectively.

Plans with 7 runs : A plan, say $A_7(1)$ may be obtained by putting $s = 7, p = 6, q = 2$ in (3.9). Another plan is displayed below.

$$A_7(2) = \rho(7; \{3^3\}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 1 & 2 & 2 & 0 \end{bmatrix}.$$

Another plan, say $A_7(3) = \rho(7; \{4^2\})$ may be obtained by taking the first two rows and the columns numbered 2,3,7,8,10,12, and 13 from the array R^7 in (3.17).

In the next section we use suitable arrays from the list above to replace one or more rows of existing orthogonal arrays and obtain inter-class orthogonal MEPs. Before that we compare the two replacing arrays $A_4(1)$ and $A_4(2)$ regarding the precision of the BLUEs of the main effect contrasts. We first compute the C-matrices (the coefficient matrices). $C_{AA;\bar{A}}$ denotes the coefficient matrix of the system of reduced normal equations for factor A . [See Notation 4.3 and (c) of Corollary 4.1].

It is rather surprising that $C_{BB;\bar{B}}$ is the same for both the plans. $C_{BB;\bar{B}} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$. $C_{AA;\bar{A}}$ is, however different in the two plans. They are given below.

$$C_{AA;\bar{A}} \text{ for } A_4(1) \text{ is } \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix} \text{ and } C_{AA;\bar{A}} \text{ for } A_4(2) \text{ is } \begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1/2 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}.$$

Remark 3.3: We note that both contrasts of the three-level factor A are estimated with the same precision in $A_4(1)$. In $A_4(2)$, however, the contrast $2\hat{\alpha}_0 - \hat{\alpha}_1 - \hat{\alpha}_2$ orthogonal to the BLUEs of the contrasts of B and hence is estimated with maximum possible precision (given the replication vector), while the contrast $\hat{\alpha}_1 - \hat{\alpha}_2$ is estimated with much less precision. Thus, while replacing a four-level factor the experimenter may choose between $A_4(1)$ and $A_4(2)$ depending on whether equal importance is attached to both the contrasts or not.

3.1 Some series of inter-class orthogonal main effect plans

Our starting point is an $OA(n, m, s, t)$ (see Definition 3.2).

Theorem 3.1 (a) Whenever an $OA(n, m, \prod_{i=1}^m s_i, 2)$ exists, an inter-class orthogonal MEP $\rho(n, 2m; \prod_{i=1}^m \{(s_i - t_i), (t_i + 1)\})$ exists. Here t_i is an integer, $1 \leq t_i \leq s_i - 2$.

(b) These inter-class orthogonal MEPs may be constructed so as to satisfy partial orthogonality property among the members of the same class similar to the description in Remark 3.1.

Proof : (a) For every $i, 1 \leq i \leq m$, one can choose a $2 \times s_i$ array, say ρ_i , with $p = s_i - t_i$ symbols in the first and $q = t_i + 1$ symbols in the second row. Now ρ_i may be used as a replacing array for the i th factor of the given OA.

(b) In particular, if ρ_i has the same structure as $A_s(1)$, with $s = s_i, p = s - t_i, q = t_i + 1$, then the members of the i th class will satisfy the stated partial orthogonality property.

Theorem 3.2 Suppose $s = 3, 4, 5$ or 7 . Whenever an $OA(n, m, s, 2)$ exists, the following series of inter-class orthogonal MEPs ρ_1 exist. Here p, q, r, s, t are nonnegative integers.

$$\rho_1 = \begin{cases} \rho(n; 3^p \times \{2^2\}^q), & p + q = m, & \text{if } s=3 \\ \rho(n; 4^p \cdot \{3.2\}^q \cdot 2^{3t}), & p + q + t = m, & \text{if } s=4 \\ \rho(n; 5^p \cdot \{4.2\}^q \cdot \{3^2\}^r \cdot \{3.2^2\}^t \cdot \{2^4\}^u), & p + q + r + t + u = m, & \text{if } s=5 \\ \rho(n; 7^p \cdot \{6.2\}^q \cdot \{4^2\}^r \cdot \{3^3\}^t), & p + q + r + t = m, & \text{if } s=7 \end{cases} \quad (3.10)$$

Proof: Let O be an $OA(n, m, s, 2)$. We keep p (out of m) of the factors of O as they are, replace every other factor by a class of factors related through an appropriate replacing array. This replacing array can be (i) an OA if it is available (which is the case when $s = 4$), (ii) one of the replacing arrays shown above or (iii) a replacing array of similar type, for instance $A_3(1)$, obtained by putting $s = 3$ in $A_s(1)$. Corollary 3.1 implies that the MEP thus constructed satisfies the required property. \square

Discussion : 1. While applying Theorem 3.2 with $s = 4$, the experimenter has a choice between the replacing arrays $A_4(1)$ and $A_4(2)$. Remark 3.3 may be useful in making the choice.

2. Comparing an inter-class(2) orthogonal MEP, say ρ_1 constructed in Theorem 3.2 with $s = 4$ with an existing plan with the same number of runs, we find the following. In the plan $ICA(n, 3^l, 2^{n-2-2l})$ of Huwang, Wu and Yen (2002), a three-level factor is orthogonal to every two-level factor and non-orthogonal to every other three-level factor [see p-349, line 7 of HWY]. In ρ_1 every three-level factor is orthogonal to every other three-level factor and all but one two-level factors, (with which it is partially orthogonal in case $A_4(2)$ is used).

3.2 More series of inter-class (2) orthogonal main effect plans

We shall now present a two-stage construction. In the first stage we start with an existing MEP, fix a subset (say \mathcal{R}) of factors and obtain a number of MEPs by replacing each factor in \mathcal{R} by a class of factors. Here we may use different replacing arrays for the same factor while constructing the first stage MEPs.

In the next stage we juxtapose these the first stage MEPs in a suitable manner to form an array. In order that the resultant array is a meaningful MEP, the replacing arrays need to satisfy certain condition as we shall see now.

Definition 3.3 Consider an MEP ρ . Let \mathcal{F} denote the class of all factors of ρ . Suppose $\rho_P(1)$ and $\rho_P(2)$ denote two replacing arrays for a factor P . If these replacing arrays have the same number of factors and the same number of levels for the corresponding factors, then they are said to be compatible. Further both of them are said to represent the same class of factors, say G_P .

Let ρ_1 and ρ_2 be two MEPs obtained from ρ by replacing the factors in a certain subset \mathcal{R} of \mathcal{F} . ρ_1 and ρ_2 are said to be compatible w.r.t. the factor P , if the corresponding replacing arrays $\rho_P(1)$ and $\rho_P(2)$ for P are compatible, in which case we say that both ρ_1 and ρ_2 are obtained by replacing P with the same class G_P of factors.

If ρ_1 and ρ_2 are compatible w.r.t. each factor in \mathcal{R} then they are said to be compatible w.r.t \mathcal{R} .

The following results are immediate from the definition.

Lemma 3.2 Consider an MEP ρ and a subset \mathcal{R} of \mathcal{F} . Suppose for every factor P in \mathcal{R} of ρ there is a class of replacing arrays $\rho_P(i, j), j = 1, 2, \dots, J, i = 1, 2, \dots, I$, such that the replacing arrays $\rho_P(i, j), j = 1, 2, \dots, J$ are mutually compatible for every $i = 1, 2, \dots, I$ and every P in \mathcal{R} . For $j = 1, 2, \dots, J$ let $G_P(i)$ denote the class of factors of each $\rho_P(i, j), j = 1, 2, \dots, J$.

Now, for every $i = 1, 2, \dots, I, j = 1, 2, \dots, J$, we obtain an array ρ_{ij} by replacing a factor P of ρ by a class of factors $G_P(i)$ related through $\rho_P(ij)$. Let

$$\rho^* = ((\rho_{ij}))_{1 \leq i \leq I, 1 \leq j \leq J}. \quad (3.11)$$

Then ρ^* represents an MEP satisfying the following.

(a) ρ^* can be viewed as an MEP directly obtained from ρ by replacing every P in \mathcal{R} with a class G_P of factors related through the following array.

$$\rho_P^* = ((\rho_P(ij)))_{1 \leq i \leq I, 1 \leq j \leq J}. \quad (3.12)$$

(b) If P and Q are mutually orthogonal in the original plan ρ , then every factor in the class $G_P(i)$ is orthogonal to every factor in the class $G_Q(i')$ in the derived plan ρ^* , $i, i' = 1, 2, \dots, I$.

(c) Fix a factor P of ρ . Fix $i \neq i', i, i' = 1, 2, \dots, I$. let

$$\rho_P(i, i') = \begin{bmatrix} \rho_P(i1) & \cdots & \rho_P(i, J) \\ \rho_P(i'1) & \cdots & \rho_P(i'J) \end{bmatrix}.$$

Let us look at the set of factors $G_P(i) \cup G_P(i')$ of the derived plan ρ^* . Two factors in this class are mutually orthogonal if and only if the corresponding factors in the plan represented by $\rho_P(i, i')$ are so.

Remark 3.4: In our definition of replacing arrays we have used the condition (3.6), so that they are not supersaturated and hence the resultant MEPs are also not supersaturated. However, we now relax this condition a bit. That is, we make use of one or more supersaturated replacing arrays in the intermediate stage, but the final MEP will not be supersaturated.

Lemma 3.3 Consider a set up just like that in the statement of Lemma 3.2, except the following. There is a factor P and an i , say i_0 , such that $\rho_P(i_0j)$ is supersaturated, i.e. it does not satisfy (3.6) for every $j = 1, \dots, J$.

Let ρ^* and ρ_P^* be as in Lemma 3.2. Let G_P denote the class of factors $G_P = \bigcup_{i=1}^I G_P(i)$.

Then, statements (a) and (b) of Lemma 3.2 hold. Further, the following modified form of Statement (c) of the same Lemma hold.

(c)' If ρ_P^* [see (3.12)] is not supersaturated, then

(i) in ρ^* any pair of factors in G_P are mutually orthogonal if and only if the corresponding factors in the plan represented by ρ_P^* are so and

(ii) the main effect contrast of each member of G_P can be estimated.

We now apply the technique of two stage construction to construct more inter-class orthogonal MEPs with two or three levels. Some of them turn out to be (fully) orthogonal.

Theorem 3.3 (a) The existence of an $OA(n, m, s, 2)$, $s = 4, 5$ or 7 implies the existence of the following inter-class orthogonal MEP.

$$\rho_2 = \begin{cases} \rho(2n, 4m; \{3^2\}^m, 2^{2m}) & \text{if } s=4 \\ \rho(2n, 4m; \{3^4\}^m) & \text{if } s=5 \\ \rho(2n, 4m; \{4^4\}^m) & \text{if } s=7 \end{cases} \quad (3.13)$$

Further, this MEP satisfies the following properties.

(b) In the case $s = 4$, the pairs of three-level factors are partial orthogonal to each other - in fact the relation between the pairs of three-level factors is just like the factors A and B of $A_8(1)$ [See Section 1].

so that every contrast is orthogonal to all except possibly another contrast. In particular, the relation between the

(c) In the cases $s = 5$ and $s = 7$, among the four members in the same class, every pair among the last 3 are mutually orthogonal through the first one.

Proof : Fix $s \in \{4, 5, 7\}$ Let $R^s(4 \times 2s)$ denote a suitably chosen array, which is partitioned as

$$R^s = ((R_{ij}))_{1 \leq i, j \leq 2}, \text{ each } R_{ij} \text{ is } 2 \times s.$$

Let O denote the given OA. We first construct four arrays $\rho_{11}, \rho_{12}, \rho_{21}$ and ρ_{22} following the method of Theorem 3.2. In this process we use R_{ij} as the replacing array for each factor to construct $\rho_{ij}, i, j = 1, 2$. Now we form

$$\rho^* = ((\rho_{ij}))_{1 \leq i, j \leq 2},$$

which is the required MEP. By Lemma 3.2, it follows that ρ^* may be viewed as the plan obtained by replacing every factor P by the class G_P of four factors related through the replacing array R^s . The rest of the proof follows from the structures of $R^s, s = 4, 5, 7$, shown below.

$$R^4 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad (3.14)$$

$$R^5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 2 & 2 & 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 & 2 & 0 & 0 & 1 & 1 & 2 \end{bmatrix} \quad (3.15)$$

$$R^7 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 1 & 1 & 3 & 3 & 1 & 2 \\ 0 & 1 & 0 & 2 & 2 & 0 & 1 & 0 & 1 & 3 & 1 & 3 & 2 & 1 \\ 0 & 1 & 2 & 0 & 2 & 0 & 0 & 1 & 1 & 3 & 3 & 1 & 1 & 2 \end{bmatrix} \square \quad (3.16)$$

Our next result is based on the elegant plan of Stark (1964), which is quoted below. [See Dey (1985), for instance, for an explicit presentation of the plan and more details].

Theorem 3.4 (Stark (1964)) *An OMEP for a 3^7 experiment on 16 runs exists.*

Theorem 3.5 (a) *The existence of an $OA(n, m, 8, 2)$ implies the existence of an orthogonal MEP for $7m$ three-level factors on $2n$ runs.*

(b) *The existence of an $OA(n, m, 4, 2)$ implies the existence of an orthogonal MEP for $6m$ three-level factors on $4n$ runs.*

Proof: Let R be a 7×16 array with symbols 0, 1, 2 representing the OMEP of Stark.

(a) We partition R as $R = [R_1 \ R_2]$, each R_i is of order 7×8 .

We first construct arrays ρ_j from the given OA by using replacing array R_j for every factor following the method of Theorem 3.2 : $j = 1, 2$. Then we form the required plan ρ^* as

$$\rho^* = [\rho_1 \ \rho_2].$$

That ρ^* satisfies the required property follows from Lemma 3.3.

(b) Let \tilde{R} denote the 6×16 array obtained by deleting a row (say the 0th one) from R . Now we partition \tilde{R} as follows.

$$\tilde{R} = ((\tilde{R}_{ij}))_{1 \leq i, j \leq 4},$$

such that \tilde{R}_{ij} is of order 2×4 for $i = 1, 2$ and 1×4 for $i = 3, 4$.

Let A denote the given OA. We construct array ρ_{ij} from A by using replacing array \tilde{R}_{ij} for every factor following the method of Theorem 3.2, $i, j = 1, 2, 3, 4$. Then we form the required plan ρ^* as

$$\rho^* = ((\rho_{ij}))_{1 \leq i, j \leq 4}.$$

Note that this procedure may be viewed as follows. Fix a factor, say P of A . The intermediate arrays $\rho_{ij}, 1 \leq j \leq 4, i = 1, 2$ are formed by replacing every factor P of A by two three-level factors each, so that $\rho_{ij}, 1 \leq j \leq 4, i = 1, 2$ are supersaturated. However, in each of the intermediate arrays $\rho_{ij}, 1 \leq j \leq 4, i = 3, 4$, P is replaced by one three-level factor. This fact, together with the choice of replacing arrays imply that the class G_P (the class of factors in ρ^* replacing P) is nothing but a class of six three-level factors, related through \tilde{R} . The rest follows from Lemma 3.3 and the fact that there are m factors in A . \square

4 Analysis of a general main effect plan.

The crucial component of data analysis of a general factorial experiment is, of course, the computation of the error sum of squares. We proceed towards a user-friendly formula for computing SS_E . The results are not new, but are not available in the form presented here. We denote the factors by F_1, F_2, \dots , instead of A, B, \dots for the sake of notational simplicity.

We assume an additive, fixed effects, main effects model with homoscedastic and uncorrelated errors having constant variance σ^2 . $\mathbf{1}_n$ will denote the $n \times 1$ vector of all-ones, while $\mathbf{J}_{m \times n}$ will denote the $m \times n$ matrix of all-ones.

Let ρ denote a main effect plan on n runs with factors F_1, F_2, \dots, F_m , F_i having a_i levels, $i = 1, \dots, m$. Let the unknown effect of the j th level of the factor F_i be denoted by α_j^i and let the $a_i \times 1$ vector α^i denote the vector of unknown effects of F_i , $1 \leq i \leq m$. Let Y_u denote the yield from the u th run, $u = 1, 2, \dots, n$. Then, assuming that in the u th run the factor F_i is set at level $l_i = l_i(u)$, $i = 1, \dots, m$ and denoting the general effect by μ , Y_u is given by

$$Y_u = \mu + \sum_{i=1}^m \alpha_{l_i}^i + \epsilon_u, u = 1, 2, \dots, n.$$

Viewing the general effect as the $(m+1)$ -th factor (F_{m+1}) and therefore writing $\alpha^{m+1} = \mu$ we express the model in matrix form as

$$\mathbf{Y} = \mathbf{X}\beta, \text{ where } \mathbf{X} = [\mathbf{X}_1 \quad \dots \quad \mathbf{X}_{m+1}] \text{ and } \beta = [\alpha^1 \quad \dots \quad \alpha^{m+1}]^T. \quad (4.17)$$

Here, X_i , the **design matrix** for F_i is a 0 – 1 matrix - the (u, t) th entry of \mathbf{X}_i is 1 if in the u th run the factor F_i is set at level t and 0 otherwise, $i = 1, 2, \dots, m$ and $\mathbf{X}_{m+1} = \mathbf{1}_n$.

Let \mathbf{T}_i denote the vector of raw totals of F_i , $i = 1, \dots, m+1$. Thus, \mathbf{T}_{m+1} is the grand total and will sometimes be denoted by G .

Notation 4.1 For any $m \times n$ matrix A , $\mathcal{C}(A)$ will denote the column space of A . Further, P_A will denote the projection operator on the column space of A . In other words, $P_A = A(A'A)^{-}A'$, where B^{-} denotes a g -inverse of B .

Notation 4.2 Let $I = \{1, 2, \dots, m+1\}$ and $S = \{i, j, \dots\}$ be a subset of I . For the sake of compactness, we introduce the following notation.

(a) $\bar{i} = I \setminus \{i\}$.

(b) $\mathbf{X}_S = [\mathbf{X}_i \quad \mathbf{X}_j \quad \dots]$.

(b) $\alpha^S = [\alpha^i \quad \dots \quad \alpha^j]^T$.

(c) \mathbf{P}_i will denote the projection operator onto the column space of \mathbf{X}_i , $i \in I$. Further, \mathbf{P}_S will denote the projection operator onto the column space of \mathbf{X}_S .

The system of reduced normal equations for a class of factors.

Notation 4.3 (a) Let S, T, U be three subsets of I such that

(i) $S \cap U = T \cap U = \phi$ and

(ii) either $S = T$ or $S \cap T = \phi$.

Let us define the matrix $\mathbf{C}_{S,T;U}$ and the vector $\mathbf{Q}_{S;U}$ as follows.

$$\mathbf{C}_{S,T;U} = ((\mathbf{C}_{ij;U}))_{i \in S, j \in T}, \mathbf{C}_{ij;U} = \mathbf{X}_i'(I - P_U)\mathbf{X}_j, \quad (4.18)$$

$$\mathbf{Q}_{S;U} = ((\mathbf{Q}_{i;U}))_{i \in S}, \mathbf{Q}_{i;U} = \mathbf{X}_i'(I - P_U)\mathbf{Y}. \quad (4.19)$$

(b) In particular, if S and T are a singleton sets, say $S = \{i\}$ and $T = \{j\}$, then we may write $\mathbf{C}_{ij;U}$ and $\mathbf{Q}_{i;U}$ instead of $\mathbf{C}_{S,T;U}$ and $\mathbf{Q}_{S;U}$ respectively. Sometimes we may write $C_{i;U}$ instead of $\mathbf{C}_{ii;U}$.

(c) Suppose $U = \{k, m+1\}$. Then we may and do write $\mathbf{C}_{S,T;k}$ and $\mathbf{Q}_{S;k}$ instead of $\mathbf{C}_{S,T;U}$ and $\mathbf{Q}_{S;U}$ respectively. [This is because $\mathbf{P}_U = \mathbf{P}_k$].

The following well-known result is presented using the notation above.

Lemma 4.1 *Suppose I is partitioned into two subsets S and U . Then, the reduced normal equations for $\widehat{\alpha^S}$, after eliminating $\widehat{\alpha^U}$ is given by*

$$C_{S,S;U}\widehat{\alpha^S} = Q_{S;U},$$

where $C_{S,S;U}$ and $Q_{S;U}$ are as given in (4.18) and the next equation.

Remark 4.1: In order that every main effect contrast of F_i is estimable, rank of $C_{i;\bar{i}}$ must be $a_i - 1$, as we know. Thus, before using a general m-factor MEP ρ with $m \geq 3$, one has to check whether $\text{Rank}(C_{i;\bar{i}}) = a_i - 1$, for every $i = 1, 2, \dots, m$.

In view of Remark 4.1 above, we define a class of MEPs, borrowing a term from the theory of block designs.

Definition 4.1 *An m-factor MEP is said to be ‘connected’ if $\text{Rank}(C_{i;\bar{i}}) = a_i - 1$, for every $i = 1, 2, \dots, m$.*

Henceforth, the MEP ρ under consideration will be assumed to be connected. We now present a few special cases of Lemma 4.1

Corollary 4.1 (a) *Consider a factor, say F_i . Let*

$\bar{i} = I \setminus \{i\}$. Then the BLUE of the main effect contrast $l'\alpha^i$ (in case it is estimable) of F_i is $l'\widehat{\alpha^i}$, where $\widehat{\alpha^i}$ is a solution of

$$C_{i;\bar{i}}\widehat{\alpha^i} = Q_{i;\bar{i}}.$$

Here the expressions for $C_{i;\bar{i}}$ and $Q_{i;\bar{i}}$ are obtained from (b) of Notation 4.3.

(b) *In particular, suppose $m = 1$. Then, the reduced normal equation for α^1 (obtained by eliminating only $F_2 = \mu$) is*

$$(\mathbf{R}_1 - \mathbf{r}_1(\mathbf{r}_1)'/n)\widehat{\alpha^1} = \mathbf{T}_1 - \mathbf{r}_1 G/n. \quad (4.20)$$

(c) *Suppose $m = 2$. Then the reduced normal equation for α^1 (obtained by eliminating F_2 and $F_3 = \mu$) is $C_{1;2}\widehat{\alpha^1} = Q_{1;2}$, where*

$$C_{1;2} = \mathbf{R}_1 - N_{12}(R_2)^{-1}N_{21} \text{ and } Q_{1;2} = \mathbf{T}_1 - N_{12}(R_2)^{-1}\mathbf{T}_2. \quad (4.21)$$

Notation 4.4 *We now define sum of squares for one or more factors, adjusted for one or more other factors. Fix a set of factors T of I . For i not in T , we define $SS_{i;T}$, the sum of squares for F_i , adjusted for the factors $F_t, t \in T$. More generally, for S disjoint from T , we define $SS_{S;T}$, the sum of squares for the set of factors $F_i, i \in S$, viewed as a single factor, adjusted for the factors $F_t, t \in T$.*

$$\begin{aligned} SS_{i;T} &= Q'_{i;T}(C_{i;T})^{-1}Q_{i;T} \\ \text{and } SS_{S;T} &= Q'_{S;T}(C_{S;T})^{-1}Q_{S;T} \end{aligned}$$

Remark 4.2: Consider two sets of disjoint factors S and T . We may view all the factors in S combined together as a single factor, say F_S , having design matrix X_S . Similarly F_T is the set of all factors in T . Then $SS_{S;T}$ may be viewed as the sum of squares for F_S adjusted for F_T .

In order to study the relationship between the sums of squares, we need the following results on partition matrices.

Lemma 4.2 *Consider a matrix W partitioned as $\begin{bmatrix} U & V \end{bmatrix}$. Let $Z = (I - P_V)U$. Then, $P_W - P_V = P_Z$.*

Corollary 4.2 Let $T \subset I$ and $i \in I \setminus T$. Let $D = (I - P_T)X_i$. Then,

$$P_D = P_{T*} - P_T, \text{ where } T* = T \cup \{i\}.$$

We need some more notation.

Notation 4.5 (a) The total sum of squares and the error sum of squares will be denoted by SS_{tot} and SS_E respectively.

(b) Fix a factor $F_i, 1 \leq i \leq m$. Let $T = \{i + 1, \dots, m + 1\}$ and $\bar{i} = I \setminus \{i\}$.

(i) Let $SS_{i;all>} = SS_{i;T}$ and

(ii) $SS_{i;all} = SS_{i;\bar{i}}$.

Thus, $SS_{i;all>}$ is the sum of squares for F_i , adjusted for the factors F_{i+1}, \dots, F_{m+1} , while $SS_{i;all}$ denotes the sum of squares for F_i , adjusted for all other factors.

Remark 4.3 : Note that $SS_{m;all>}$ is the so-called unadjusted sum of squares for F_m .

We are now in a position to present the computational formulae of the error sum of squares.

Theorem 4.1 Consider a main effect plan with m mutually non-orthogonal factors F_1, F_2, \dots, F_m . The error sum of squares (SS_E) may be computed from the total sum of squares (SS_{tot}) as follows.

$$SS_E = SS_{tot} - SS_{sub}, \text{ where} \quad (4.22)$$

$$SS_{sub} = \sum_{i=1}^m SS_{i;all>}. \quad (4.23)$$

Theorem 4.2 The data obtained from a connected main effect plan with m mutually non-orthogonal factors may be analyzed using the following table.

Table 2.1 : ANOVA for an m -factor non-orthogonal main effect plan

| Source | d.f | S.S adjusted for all others | S.S adjusted for the next ones | F-statistics |
|------------------|-----------|--------------------------------|--------------------------------------|-------------------------------------|
| F_1 | $a_1 - 1$ | $SS_{1;all}$ | $SS_{1;all>} = SS_{1;2,\dots,m}$ | $\frac{SS_{1;all}/(a_1-1)}{SS_E/e}$ |
| F_2 | $a_2 - 1$ | $SS_{2;all}$ | $SS_{2;all>} = SS_{2;3,\dots,m}$ | $\frac{SS_{2;all}/(a_2-1)}{SS_E/e}$ |
| \vdots | \vdots | \vdots | \vdots | |
| F_m | $a_m - 1$ | $SS_{m;all}$ | $SS_{m;all>} = SS_{m;m+1}$ | $\frac{SS_{m;all}/(a_m-1)}{SS_E/e}$ |
| To be subtracted | - | - | $SS_{sub} = \text{Sum of all above}$ | |
| Error | e | $SS_E =$ | $SS_{tot} - SS_{sub}$ | |
| Total | $n-1$ | $SS_{tot} =$ | $\sum_{u=1}^n Y_u^2 - G^2/n$ | |

Here the error degrees of freedom is $e = n - 1 - \sum_{i=1}^m (a_i - 1)$, as usual.

Extension to a general factorial experiment: Consider a plan for a factorial experiment with k factors. Let E denote a factorial effect - a main effect or a t -factor interaction, $2 \leq t \leq k$. We list the factorial effects under study as say E_1, \dots, E_m , where m is the number of factorial effects of interest. Then we treat these E_i 's in the same way as the main effects F_i 's are treated above. That is we denote the design matrix and the unknown effects of E_i as X_i and α^i as here, orders of these would be different when the effects are interactions. Thus, following the same argument, we arrive at the following result.

Theorem 4.3 *The error sum of squares of a general factorial experiment can be obtained in the same manner as described in Theorem 4.1.*

Situations when analysis is considerably simpler.

We have seen in Theorem 4.1 that analysis of a general MEP is rather involved - needs computation of $2m - 1$ sums of squares for an m -factor plan. We now look for situations when so much computation is not needed.

We know that when there is only one treatment factor F_1 , the sum of squares for F_1 is nothing but the so-called unadjusted sum of squares $T_1'(R_1)^{-1}T_1 - G^2/n$. Moreover, in the situations when there are two factors, say F_1 and F_2 , the sum of squares for F_1 (obtained by adjusting for F_2) is $SS_{1;2} = Q'_{1;2}(C_{1;2})^{-1}Q_{1;2}$ where $C_{1;2}$ and $Q_{1;2}$ are as in (4.21). (See (b) and (c) of Corollary 4.1).

Now we seek the answer to the following questions. **Consider a main effect plan for m factors ($m \geq 3$). Fix a factor, say F_i . What conditions must the design matrices satisfy so that the sum of squares for F_i adjusted for all others is the same as**

(a) **the unadjusted sum of squares for F_i ?**

(b) **the sum of squares for F_i adjusted for only one factor, (say F_m) ?**

[That is so far as F_i is concerned, other factors are virtually absent.]

Theorem 4.4 *Fix a factor, say F_i .*

(a) *A necessary and sufficient condition for $SS_{i;all} = SS_{i;m+1}$ is that the incidence matrix N_{ij} satisfies the proportional frequency condition stated in (2.1) [see Definition 2.3.*

(b) *A necessary and sufficient condition for $SS_{i;all} = SS_{i;m}$ is that*

$$N_{ij} = N_{im}(R_m)^{-1}N'_{jm}, j \neq i, 1 \leq i, j \leq m-1. \quad (4.24)$$

The proof relies on two lemmas we present now.

Lemma 4.3 *Consider matrices $A(m \times n), B((m \times p))$ such that*

$$\mathcal{C}(B) \subseteq \mathcal{C}(A).$$

Let $C((m \times q))$ be any matrix. Then a necessary and sufficient condition that $\mathcal{C}(P_B C) = \mathcal{C}(P_A C)$ is that $(P_A - P_B)C = 0$.

Lemma 4.4 *Consider a matrix W partitioned as $\begin{bmatrix} U & V \end{bmatrix}$. Let $Z = (I - P_V)U$. Then, $P_W - P_V = P_Z$.*

Proof of theorem 4.4: Let $T = \{1, 2, \dots, i-1, i+1, \dots, m\}$ and $T^* = T \cup \{m+1\}$. From Notation 4.5 (b), we see that

$$SS_{i;all} = Y'P_U Y, \quad SS_{i;m+1} = Y'P_V Y,$$

where $U = (I - P_{T^*})X_i$ and $V = (I - P_{m+1})X_i$.

Proof of (a); From the expressions above, a necessary and sufficient condition for $SS_{i;all} = SS_{i;m+1}$ is that $P_U = P_V$, that is $\mathcal{C}(U) = \mathcal{C}(V)$. Take $A = X_{m+1}$, $B = X_{T^*}$, $C = X_i$. Then, clearly, $\mathcal{C}(A) \subset \mathcal{C}(B)$, that is $[\mathcal{C}(B)]^\perp \subset [\mathcal{C}(A)]^\perp$. By Lemma 4.3 a necessary and sufficient condition for $\mathcal{C}(U) = \mathcal{C}(V)$ is that $[(I - P_A) - (I - P_B)]C = 0$, which is same as

$$(P_{T^*} - P_{m+1})X_i = 0. \quad (4.25)$$

Now by Lemma 4.4, $P_{T^*} - P_{m+1} = P_Z$, where $Z = (I - P_{m+1})X_T$. Thus, (4.25) is

$$\Leftrightarrow P_Z X_i = 0 \Leftrightarrow X'_i Z = 0 \Leftrightarrow X'_i (I - P_{m+1}) X_j = 0, j \neq i,$$

which is the same as the proportional frequency condition.

Proof of (b) : Proceeding along similar lines as in the proof of (a), we find that the necessary and sufficient condition for $SS_{i;all} = SS_{i;m}$ is that

$$P_W X_i = 0, \text{ where } W = (I - P_m)X_T. \quad (4.26)$$

But this condition $\Leftrightarrow X'_i W = 0 \Leftrightarrow X'_i (I - P_m) X_j = 0, j \neq i, 1 \leq i, j \leq m-1$. This condition simplifies to the form in the statement. \square

Remark 4.4 : The sufficiency part of (a) of Theorem 4.4 is well-known. We now point out that the respective conditions are also necessary for the sum of squares to satisfy these desirable properties.

Properties of a plan orthogonal through a factor.

Let us recall Definition 2.5.

Theorem 4.5 *An MEP orthogonal through F_m has the following properties.*

(a) *For every factor $F_i, 1 \leq i \leq m-1$, the reduced normal equation for $\hat{\alpha}^i$ is*

$$(\mathbf{R}_i - \mathbf{N}_{im}(R_m)^{-1}(\mathbf{N}_{im})') \hat{\alpha}^i = T_i - \mathbf{N}_{im}(R_m)^{-1}T_m.$$

(b) *The error sum of squares is obtained by subtracting the following from the total sum of squares. Add the sum of squares for each F_j adjusted for $F_m, 1 \leq j \leq m-1$ and then the unadjusted sum of squares for F_m . Symbolically,*

$$SS_E = SS_{tot} - \sum_{j=1}^{m-1} SS_{j;m} - SS_{m;m+1}.$$

Proof : Let $L = \{1, 2, \dots, m-1\}$. Then, the reduced normal equation for the combined effect of the vector of treatment factors $(\hat{\alpha}^L)$ (after eliminating $\hat{\mu}$ and $\hat{\alpha}^m$) is

$$C_{LL;m} \hat{\alpha}^L = Q_{L;m}, \text{ where } C_{LL;m} = ((C_{ij;m}))_{1 \leq i, j \leq m-1}, C_{ij;m} = X'_i (I - P_m) X_j, \quad (4.27)$$

$$Q_{L;m} = ((Q_{i;m}))_{1 \leq i \leq m-1}, Q_{i;m} = X'_i (I - P_m) Y. \quad (4.28)$$

(a) Since the plan is orthogonal through F_m , $C_{ij;m} = 0, i \neq j, 1 \leq i, j \leq m-1$. Thus, the reduced normal equation for $\hat{\alpha}^i$ is $C_{ii;m} \hat{\alpha}^i = Q_{i;m}$, where $C_{ii;m}$ and $Q_{i;m}$ are as in (4.27) and the next equation. That $C_{ii;m}$ and $Q_{i;m}$ are of the form in the statement of the theorem can be verified easily.

(b) Since the off-diagonal block matrices of $C_{LL;m}$ are null,

$$SS_{L;m,m+1} = \sum_{i=1}^{m-1} Q'_{i;m} (C_{ii;m})^{-1} Q_{i;m} = \sum_{i=1}^{m-1} SS_{i;m}$$

Now the rest follows from (4.22). \square

Remark 4.5: Statement (a) of Theorem 4.5 was observed as early as 1996 by Morgan and Uddin in their Theorem 2.1 [equation (7)]. However, since their paper essentially was concerned with the construction of optimal nested row-column designs, the result was overlooked by many authors (such as Mukherjee, Dey and Chatterjee (2001) and Bagchi (2010)) working on blocked main effect plans.

Remark 4.6: (b) of Theorem 4.5 shows that a plan orthogonal through one factor (F_m) considerably simplifies the computation of error SS as well as the sum of squares for the treatment factors $F_1, F_2 \dots F_{m-1}$. Thus, in case F_m happens to be a block factor, the whole analysis is only a little more involved than a fully orthogonal plan, as has been noted in Bagchi (2010). However, in the situation when F_m is a treatment factor, analysis is a little more involved since $SS_{m;all}$ needs to be computed. Needless to mention that the precision of the BLUEs of the main effect contrasts of F_m (being non-orthogonal to $m-1$ other factors) is less than the other factors.

Remark 4.7: Let us recall the plan $A_5(2)$ [see (2.5)]. If we remove the last column (run) and the last row (factor D), then we get an OA (4,3,2,2), say ρ^* . Let C_Q denote the coefficient matrix of the reduced normal equation for factor Q, $Q = A, B, C$ obtained from the plan ρ^* . One may check that $C_{Q;Q} = C_Q$ for $Q = A, B, C$. Thus, even though $A_5(2)$ is not orthogonal, the main effects of factors A, B and C are estimated with the same precision as the orthogonal plan ρ^* . Therefore, by adding one more run, we are able to accommodate one more factor (D), without sacrificing the precision of the three existing factors. The main effect of D is, however, being estimated with less precision than the others.

Data analysis of an inter-class orthogonal plan.

Notation 4.6 Consider an inter-class orthogonal plan with k classes, the i th class having m_i factors denoted by $F_{i,1}, \dots, F_{i,m_i}$, $1 \leq i \leq k$. Let F_G denote the general effect.

(a) Let α^{ij} and X_j^i denote respectively the vector of unknown effects and the design matrix of F_{ij} , $1 \leq j \leq m_i$, $i = 1, \dots, k$.

(b) Let $I_i = \{(i, 1), \dots, (i, m_i)\}$. For a fixed j , $1 \leq j \leq m_i$, let $T_j = \{(i, j+1), \dots, (i, m_i)\}$ and $\bar{j} = I_i \setminus \{(i, j)\}$.

(c) $SS_{j;U}^i$ will denote the sum of squares for $F_{i,j}$ adjusted for each $F_{i,k}$, $k \in U$, where j is not in U .

(d) $SS_{j;all}^i$ will denote the sum of squares for $F_{i,j}$ adjusted for all other factors in its own class, i.e. $SS_{j;all}^i = SS_{j;\bar{j}}^i$.

Further, $SS_{j;all>}^i$ will denote the sum of squares for $F_{i,j}$ adjusted for all factors next to it in its own class, $1 \leq j \leq m_i - 1$, while $SS_{m_i;all>}^i$ will denote the sum of squares for F_{i,m_i} adjusted for F_G (that is the unadjusted sum of squares). Thus,

$$SS_{j;all>}^i = SS_{j;T_j}^i, 1 \leq j \leq m_i - 1 \text{ and } SS_{m_i;all>}^i = SS_{(i,m_i);G}.$$

(e) The following expression will be referred to as the class total for the i th class.

$$SS_{total}^i = \sum_{j=1}^{m_i} SS_{j;all>}^i.$$

Theorem 4.6 Consider an inter-class orthogonal plan as in Notation 4.6. Fix a class, say the i th one and a factor, say $F_{i,j}$.

(a) The reduced normal equation for $\widehat{\alpha^{ij}}$ is obtained by eliminating only the other factors in the i th class. More explicitly, the reduced normal equation is as follows.

$$C_{j;\bar{j}}^i \widehat{\alpha^{ij}} = Q_{j;\bar{j}}^i \text{ where } C_{j;\bar{j}}^i = (X_j^i)'(I - P_{\bar{j}}^i)X_j^i \text{ and } Q_{j;\bar{j}}^i = (X_j^i)'(I - P_{\bar{j}}^i)Y.$$

Here $P_{\bar{j}}^i$ is the projection operator onto the column space of $X_{\bar{j}}^i$.

(b) The sum of squares for $F_{i,j}$, adjusted for all other factors, is nothing but the sum of squares adjusted for all other factors in the i th class. Similar statements hold for the sum of squares adjusted for all factors next to $F_{i,j}$. Symbolically,

$$SS_{(i,j);all} = SS_{j;all}^i \text{ and } SS_{(i,j);all>} = SS_{j;all>}^i.$$

(c) The error sum of squares is obtained by subtracting the class totals for all the k classes from the total sum of squares. Symbolically,

$$SS_E = SS_{tot} - \sum_{i=1}^k SS_{total}^i.$$

We now note, that if all the factors of a class except one are mutually orthogonal through that one, the computation of the class total is considerably simpler. The proof follows from Theorem 4.5.

Theorem 4.7 Suppose an inter-class orthogonal plan has a class (say the i th one) in which all the factors are orthogonal through F_{i,m_i} . Then, the class total for this class can be expressed as follows.

$$SS_{total}^i = \sum_{j=1}^{m_i-1} SS_{j;m_i}^i + SS_{m_i;G}.$$

5 Main effect plans of small size.

In this section, we present MEPs with fifteen or less runs obtained by ad-hoc methods. The factors have at most five levels and the class size of each plan is at most three. The plans having class-size three are $A_{12}(1)$, $A_{12}(3)$ and $A_{12}(4)$. Further, all plans except $A_{12}(2)$ and $A_{12}(3)$ are saturated. The graph next to each plan shows the relationship between factors : the edges drawn with continuous lines represent orthogonality while dotted line indicate partial orthogonality. The factors are named as A,B, in the natural order. The equal frequency plans are indicated by “*”.

We begin with a general plan for two p -level and one two-level factors on $2p$ runs. If $p = 3$, the levels of A form a balanced incomplete block design (BIBD) with those of B .

$$A_{2p}(1) = \rho(2p, 3; \{p^2\}.2) =$$

$$\begin{bmatrix} 0 & 1 & \cdots & p-1 & 0 & 1 & \cdots & p-1 \\ 0 & 1 & \cdots & p-1 & 1 & 2 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \end{bmatrix} \quad \begin{array}{c} \text{A} \bullet \quad \bullet \text{B} \\ \quad \diagdown \quad \diagup \\ \quad \text{C} \bullet \end{array}$$

Now a plan with **6 runs**.

$$A_6(1) = \begin{bmatrix} A & 0 & 0 & 1 & 1 & 2 & 2 \\ B & 0 & 1 & 0 & 0 & 1 & 0 \\ C & 0 & 1 & 0 & 1 & 0 & 1 \\ D & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Remark 5.1: For the same experiment an equal-frequency plan is available - plan $L_6(3.2^3)$ of Wang and Wu (1992). The graphical representation of these two plans are shown below.

$$L_6(3.2^3) = \begin{array}{c} \bullet \text{B} \\ \bullet \text{C} \\ \bullet \text{D} \\ \diagup \quad \diagdown \quad \diagup \\ \bullet \text{A} \end{array} \quad \text{while} \quad A_6(1) = \begin{array}{cc} \text{A} \bullet & \text{B} \bullet \\ \diagdown & \text{---} \diagup \\ \text{C} \bullet & \text{D} \bullet \end{array}$$

Regarding the performances, the new plan estimates all but one contrast ($C_1 = \hat{a}_0 - 2\hat{a}_1 + \hat{a}_2$) with equal or more precision. Using the formulae in Theorem 4.6, one may check that the amount of computation is also less here. However, C_1 may be more important for some experimenter, in which case the old plan $L_6(3.2^3)$ would be preferable.

We now present plans on **8 runs**. We take up the well-known $OA(8, 4, 3, 2^4)$ and add one more two-level factor (F) with unequal frequency such that it is orthogonal to all other factors except A.

(a) $A_8(2) = \rho(8, 6; \{3 \times 2\}.2^4) =$

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Our next **plan on 8 runs** has two three-level factors satisfying partial orthogonality.

(b) The plan $A_8(3) = \rho(8, 5; \{3^2\}.2^2).2 =$

$$\begin{bmatrix} 0 & 1 & 0 & 2 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

We now present two plans with 4-level factors on 8 runs. Note that on 8 runs, a four-level factor can be orthogonal to neither a four-level nor a three-level factor. Using non-orthogonality, we are able to accommodate two four-level factors in one plan and one four-level and one three-level factor in another plan on 8 runs.

(c) $A_8(4)^* = \rho(8, 3; \{4^2\}.2) =$ is obtained by putting $p = 4$ in $A_{2p}(1)$. In this plan, the 4-level factors A and B form a group divisible design ($m = n = 2, r = k = 2, \lambda_1 = 0, \lambda_2 = 1$).

(d) $A_8(5) = \rho(8, 4; \{4.3\}.2^2) =$

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

A plan on 10 runs : $A_{10}^* = \rho(10, 3; \{5^2\}.2) =$ is obtained by putting $p = 5$ in $A_{2p}(1)$. Here the 5-level factors form a symmetric cyclic PBIBD with $r = k = 2, \lambda_1 = 1, \lambda_2 = 0$.

We shall now present **plans on 12 runs** : There is no plan in the literature accommodating one or more 4-level factors on 12 runs. So, we begin with such plans.

(a) $A_{12}(1)^* = \rho(12, 5; \{2.4^2\}.3^2) =$

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 2 & 3 & 0 & 3 & 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 2 & 1 & 0 & 0 & 2 & 1 & 1 & 2 & 0 \end{bmatrix}$$

In this equal frequency saturated plan, both the four-level factors B and C form a generalized group divisible design with the levels of the two-level factor A. Between themselves, they form a Balanced incomplete block design (BIBD). The relation between factors D and E is presented in details after Remark 2.4.

$$(b) A_{12}(2) = \rho(12, 4; \{3^2\}.\{3.4\}) =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 2 & 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \end{bmatrix}$$

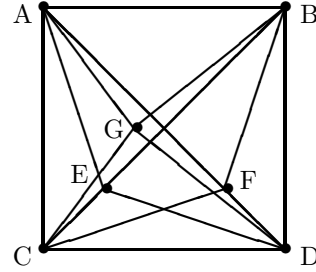


Here all the factors other except C has equal frequency. The levels of factors A and B form a balanced block design (BBD). D is partially orthogonal to C as contrast $\hat{\delta}_1 - \hat{\delta}_2$ is orthogonal the contrasts for C . However, C is non-orthogonal to D .

Remark 5.2: In the plan $A_{12}(2)$, the four-level factor D may be replaced by three mutually orthogonal two-level factors to obtain an almost orthogonal MEP for an $3^3.2^3$ experiment.

$$(c) A_{12}(3)^* = \rho(12, 7; 2^4.\{3^3\}) =$$

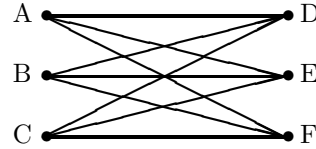
$$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 0 & 1 & 0 & 2 & 1 & 2 & 2 & 0 & 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 & 0 & 1 & 2 & 2 & 1 & 0 & 0 \end{bmatrix}$$



Remark 5.3: The plan $A_{12}(3)$ is very similar to the plan $L'_{12}(3^4.2^3)$ of Wang and Wu (1992). The difference is that $A_{12}(3)$ provides one more two-level factor and one less three-level factor and so has total d.f one less than $L'_{12}(3^4.2^3)$. On the other hand, since each three-level factor (say P) in $A_{12}(3)$ is non-orthogonal to two and not three factors and the relationship of P with the any other three-level factor is the same as that in $L'_{12}(3^4.2^3)$ it's contrasts are estimated with greater precision.

$$(d) A_{12}(4) = \rho(12, 6; \{3^3\}.\{3^2.2\}) =$$

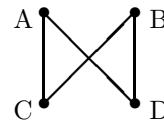
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 2 & 1 & 2 & 0 & 0 & 2 & 1 & 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 1 & 0 & 2 & 1 & 2 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 0 & 1 & 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$



This inter-class (3) orthogonal MEP has accommodated five three-level factors together with a two-level factor. The levels of A form a BBD with those of each of B and C, while the levels of B form a variance-balanced non-binary design with those of C. Both the three-level factors D and E are partially orthogonal to the two-level factor F .

$$\text{A plan on 15 runs : } A_{15} = \rho(15, 4; \{3^2\}.\{5^2\}) =$$

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 & 0 & 1 & 2 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 \\ 0 & 1 & 2 & 1 & 2 & 3 & 2 & 3 & 4 & 3 & 4 & 0 & 4 & 0 & 1 \end{bmatrix}$$



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